where $f_k^{-1}(\cdot)$ is a monotonic function in the interval $[0, 2\pi]$ and inverse of $f_k(\tau)$ in (6.6); it can be determined by using curves in Fig. 5. Having determined by formulas (6.8) the quantities T and T_1 for the specified a, we can calculated modes of time-optimum operation by formulas (6.1) – (6.3) and (2.3). The time-optimum operation mode (6.1), i.e. the solution of problem (1) which corresponds to T, is close with respect to the functional to the simpler mode (6.2) with three constant velocity sections, which corresponds to time T_1.

The maximum relative errors with respect to the functional, resulting from the substitution of the mode with three sections for the optimum one, does not exceed $|\Delta a / a| < 1.1\%$ in the case of problem (1) and $|\Delta T / T| < 1.2\%$ for problem (2) for any a and T.

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STUDY OF STEADY MODES OF DISTURBED AUTONOMOUS SYSTEMS IN CRITICAL CASES

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A method due to Poincaré is used to study the critical cases in essentially nonlinear autonomous systems with one degree of freedom, and the situations leading to the splitting of the trajectories. The first Liapunov method is used to study the problems of stability of the steady modes. A selfrotating, almost conservative system is considered as an example. Previous papers concerned with the analysis of the motions near the generating family of periodic or rotational motions of an unperturbed system dealt, as a rule, with relatively simple cases in which the equations of the parameters of the family defining the steady mode admit, in the first approximation, simple real roots /1 – 6/. Subtler and more complex cases in which the roots are multiple, or when some of the equations of the defining system are satisfied identically, were given much less attention /1, 7 – 11/.

1. Statement of the problem. We consider a wide class of autonomous systems with one degree of freedom and slowly varying parameters of the form

$$q^{\cdot} = Q(q, p, x, \varepsilon), \quad p^{\cdot} = P(q, p, x, \varepsilon), \quad x^{\cdot} = \varepsilon X(q, p, x, \varepsilon) \tag{1.1}$$

Here p and q are generalized coordinates, x is the vector of the parameters of the system and $\varepsilon \in [0, \varepsilon_0]$ is a small nonnegative parameter. The functions Q, P and X are assumed to be sufficiently smooth in a certain relevant domain of variation of their arguments, and to satisfy the necessary conditions of periodicity with respect to the rotating variables q or p. Finally, $t \geqslant t_0$ denotes time, the initial conditions are not given.

If we assume the existence of a complete family of periodic or rotational solutions at $\varepsilon = 0$, the system (1.1) can be reduced to its general standard form with a rotating phase

$$a^{\cdot} = \varepsilon f(a, \psi, \varepsilon), \quad \psi^{\cdot} = \omega(a) + \varepsilon F(a, \psi, \varepsilon) \tag{1.2}$$

Here a is a quasiconstant vector, $\omega(a)$ is the frequency of oscillations or rotations and $\infty > \omega_1 \geqslant \omega \geqslant \omega_0 > 0$. The right-hand sides are periodic in phase ψ with a constant period of 2π and are sufficiently smooth functions of their arguments for $|\psi| < \infty$ and $a \in [a_1, a_2]$ with $\max|\Delta a| \to 0$ as $\varepsilon \to 0$.

Since we shall construct the solution in the form of series, we shall simplify the matters by assuming that the right-hand sides are analytic. This assumption is not, in general, obligatory as an exact solution can be constructed by consecutive approximations in whole or fractional powers of the parameter ε without making any assumptions of analyticity /1/. We note that the smoothness requirement in the process of constructing a scheme of consecutive approximations must be strengthened in appropriate manner when dealing with the critical cases. The meaning of this remark is made clear in the constructions given in Sects. 2 and 3.

A direct construction of a solution of the system (1.2) introducing the perturbed frequency in the sequence of increasing secular terms /1, 2/, leads to very cumbersome expressions. A significant simplification is achieved by preliminary construction of a periodic phase trajectory $a(\psi, \varepsilon)$, after which the required solution is obtained as a function of time using the method of quadratures /12/.

The periodic phase trajectory is described by the following standard system:

$$\frac{da}{d\psi} = \varepsilon \frac{f(a, \psi, \varepsilon)}{\omega(a) + \varepsilon F(a, \psi, \varepsilon)} \tag{1.3}$$

Since in the region under consideration we have $\omega \geqslant \omega_0 > 0$, we find that for a sufficiently small $\varepsilon > 0$ the following one-to-one correspondence exists between ψ and t:

$$t - t_0 + \tau = \int_0^{\psi} [\omega(a(\psi', \varepsilon)) + \varepsilon F(a(\psi', \varepsilon), \psi', \varepsilon)]^{-1} d\psi' \tag{1.4}$$

If a 2π-periodic solution of (1.3) has been constructed, then the formula (1.4) can be used to find the required $T(\varepsilon)$-periodic solution of the initial system (1.2), of the form /12/

$$\psi = \frac{2\pi}{T(\varepsilon)}(t - t_0 + \tau) + \varepsilon \Psi\left(\frac{2\pi}{T(\varepsilon)}(t - t_0 + \tau), \varepsilon\right), \quad \tau = \text{const} \tag{1.5}$$

$$T = T(\varepsilon) = \int_0^{2\pi} [\omega(a(\psi, \varepsilon)) + \varepsilon F(a(\psi, \varepsilon), \psi, \varepsilon)]^{-1} d\psi \tag{1.6}$$

where Ψ is a T-periodic function of time.

The problems of the Liapunov stability of the periodic solutions of the system (1.2) can be solved by investigating the stability of the periodic phase trajectories of the system (1.3) with the help of the Andronov-Vitt theorem /1/. Namely, if all characteristic indices of the corresponding system with periodic coefficients in the variations

$$\frac{d\delta a}{d\psi} = \varepsilon \frac{\partial}{\partial a}\left[\frac{f(a, \psi, \varepsilon)}{\omega(a) + \varepsilon F(a, \psi, \varepsilon)}\right]\delta a, \quad a = a(\psi, \varepsilon) \tag{1.7}$$

have negative real parts, then the solution of the initial system (1.2) is orbitally stable.

Let us now construct a periodic solution of the system (1.3). First we briefly explain the idea of the Poincaré method of small parameter. This method of constructing a solution is based on choosing the initial condition for the quasiconstant vector a in a special manner, namely the quantity a_0 and the small increment v which vanishes when $\varepsilon = 0$ are chosen such that the conditions

$$a(\psi_0, a_0, v, \varepsilon) = a_0 + v = a(\psi_0 + 2\pi, a_0, v, \varepsilon) \tag{1.8}$$

hold. The Eqs. (1.8) represent the necessary and sufficient conditions for the periodicity of the solution $a(\psi, a_0, v, \varepsilon)$ of (1.3).

We note that the initial value ψ_0 can be chosen arbitrarily. Then the expression for the function sought

$$a(\psi, a_0, v, \varepsilon) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}(\psi, a_0)\varepsilon^i v^j, \quad a_{00}(\psi, a_0) = a_0 = \mathrm{const} \tag{1.9}$$

is substituted into (1.3). Taking (1.8) into account and equating the coefficients of like powers of $\varepsilon^i v^j$, we use the Weierstrass theorem on implicit functions /13/ to obtain an expression for the unknown small increment $v = v(\varepsilon)$. Since v is a vector, the expression (1.9) means that

$$\sum_{j=0}^{\infty} a_{ij}v^j = \sum_{j_1=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} a_{ij_1\ldots j_n} v_1^{j_1} \cdots v_n^{j_n}$$

where n denotes the dimension of the vector a. As a result, the expressions for $v(\varepsilon)$ and the solution (1.9) are obtained in the form of series in whole or fractional powers of the parameter ε.

The theoretical justification of the scheme of constructing the solution is not considered here.

2. Basic results. It was shown in /4/ that when

$$\frac{1}{\omega(a_0)} \det\left(\frac{\partial R_1(a_0)}{\partial a_0}\right)\bigg|_{a_0^*} \neq 0 \tag{2.1}$$

where a_0^* is a real root of the vector equation

$$R_1(a_0) \equiv \frac{1}{\omega(a_0)} \int_0^{2\pi} f(a_0, \psi, 0)\, d\psi = 0 \tag{2.2}$$

then a periodic solution of the system (1.3) ixists and is unique. The solution can be constructed in the form of series, or by means of successive approximations in whole powers of ε. The sufficient condition of stability is expressed by the requirement that the roots of the characterisitc equation of the first approximation

$$\det(\varepsilon(\partial R_1 / \partial a_0)^* - I\lambda) = 0 \tag{2.3}$$

have negative real parts.

The cases in which $\det(\partial R_1 / \partial a_0)^* = 0$, we shall call singular or critical ones.

A complete analysis of these cases is extremely difficult, especially when $n > 1$.

1°. Assume a to be a scalar and let a_0^* be a double real root of (2.2), belonging to the region under consideration. In this case the required periodic solution is given by the series

$$a(\psi, a_0, \varepsilon) = a_0 + \sum_{i=1}^{\infty} \delta^i a_i(\psi, a_0), \qquad \delta = \sqrt{\varepsilon} \tag{2.4}$$

Here a_i are the unknown periodic coefficients which can be found by substituting (2.4) into (1.3) and equating the coefficients of like powers of δ. This procedure leads to a linked sequence of explicitly integrable equations. In particular, we have

$$a_1 = \text{const}; \quad a_2 = a_{20} + (\omega)_0^{-1} \int_0^{\psi} (f)_0 \, d\psi, \quad a_{20} = \text{const}. \tag{2.5}$$

Here the constants of integration $a_1, a_{20}, \ldots$ can be found from the conditions of periodicity of the higher coefficients beginning from a_4. For a_1 we have the equation

$$\frac{a_1^2}{2}\left(\frac{d^2 R_1}{da_0^2}\right)^* = (\omega)_0^{-2} \int_0^{2\pi} \left[(f)_0 (F)_0 - (\omega)_0 \left(\frac{\partial f}{\partial \varepsilon}\right)_0\right] d\psi \equiv L(a_0^*) \tag{2.6}$$

Here and henceforth the expressions of the type (f_0) mean that the arguments $a = a_0^*$ and $\varepsilon = 0$. Equation (2.6) admits real roots when $L\,(d^2R_1 / da_0^2)^* \geqslant 0$. Let us assume that we have the strict inequality. Then the following relations for determining the unknown constants a_{i0} $(i \geqslant 2)$ will become linear and will contain the multiplying factor $a^*_1\,(d^2R_1 / da_0^2)^*$. In particular, for $i = 2$ we have

$$\begin{aligned}
a_{20}^* = &-\left[a_1^* \left(\frac{d^2R_1}{da_0^2}\right)^*\right]^{-1} a_1^* \int_0^{2\pi} \Bigg\{\Bigg[-(f)_0 \frac{(\omega'')_0}{(\omega)_0^2} + 2(f)_0 \frac{(\omega')_0^2}{(\omega)_0^3} - \\
&2\left(\frac{\partial f}{\partial a}\right)_0 \frac{(\omega')_0}{(\omega)_0^2} + \left(\frac{\partial^2 f}{\partial a^2}\right)_0 \frac{1}{(\omega)_0}\Bigg] c(\psi) + a_1^{*2}\Bigg[(f)_0 \frac{(\omega'')_0 (\omega')_0}{(\omega)_0^3} - \\
&(f)_0 \frac{(\omega''')_0}{6(\omega)_0^2} - (f)_0 \frac{(\omega')_0^3}{(\omega)_0^4} - \left(\frac{\partial f}{\partial a}\right)_0 \left(\frac{(\omega'')_0}{2(\omega)_0^2} - \frac{(\omega')_0^2}{(\omega)_0^3}\right) - \\
&\left(\frac{\partial^2 f}{\partial a^2}\right)_0 \frac{(\omega')_0}{2(\omega)_0^2} + \left(\frac{\partial^3 f}{\partial a^3}\right)_0 \frac{1}{6(\omega)_0}\Bigg] - \left(\frac{\partial F}{\partial a}\right)_0 \frac{(f)_0}{(\omega)_0^2} + \\
&(F)_0 (f)_0 \frac{(\omega')_0}{(\omega)_0^3} - \left(\frac{\partial f}{\partial a}\right)_0 \frac{(F)_0}{(\omega)_0^2} - \left(\frac{\partial f}{\partial \varepsilon}\right)_0 \frac{(\omega')_0}{(\omega)_0^2} + \\
&\left(\frac{\partial^2 f}{\partial a\, \partial \varepsilon}\right)_0 \frac{1}{(\omega)_0}\Bigg\} d\psi, \qquad c(\psi) = (\omega)_0^{-1} \int_0^{\psi} (f)_0 \, d\varphi
\end{aligned}$$

The expressions for the higher coefficients are very cumbersome, therefore we do not give them here. The subsequent analysis shows that the phase trajectory splits by a quantity of the order $O(\delta)$

$$a(\psi, a_0^*, \varepsilon) = a_0^* + \sum_{i=1}^{\infty} (\pm\delta)^i a_i(\psi, a_0^*)$$

If $L(a_0^*) = 0$, then all odd coefficients a_1, $a_3(\psi), \ldots$ vanish and the expansion takes place in whole powers of the parameter ε. However, in this case additional conditions of existence of a periodic solution appear, and the splitting is of the order $O(\varepsilon)$, i.e. the perturbed solution is not unique in the accepted sense /1/.

Similarly, using the Poincaré approach we can obtain sufficient conditions of existence of a periodic solution for the case of arbitrary multiplicity r. We establish that the expansions can be carried out in various fractional powers of the parameter ε, i.e. in the powers $\delta_i(r) = \varepsilon^{1/r_i}$, where r_i is an integer. We also have the inequality

$$\sum_{i=1}^{k} r_i \leqslant r$$

where k is the number of different expansions.

Using (1.7), we now obtain the approximate value of the characteristic index λ for the already considered case of $r = 2$ and $L(a_0^*) \neq 0$, i.e. for $a_1^* \neq 0$

$$\lambda = \frac{\varepsilon^{3/2}}{2\pi} a_1^* \left(\frac{d^2R_1}{da_0^2}\right)^* + O(\varepsilon^2)$$

From this it follows that for a sufficiently small $\varepsilon > 0$ one of the branches of the solution is stable, and the other is Liapunov unstable. The analysis of stability for the scalar variable a in the case when the root a_0^* is of arbitrary multiplicity r, is carried out in the analogous manner.

2°. Next we consider a simple case of higher order motions. Let a be a vector and let the system (2.2) be satisfied identically, i.e. independently of a_0. Then, using the proposed method for a_0 we obtain the following defining system of equations:

$$R_2(a_0) \equiv (\omega)_0^{-2} \int_0^{2\pi} \left\{(\omega)_0\left[\left(\frac{\partial f}{\partial \varepsilon}\right)_0 + \left(\frac{\partial f}{\partial a}\right)_0 c(\psi, a_0)\right] - (f)_0\left[(\omega)_0' c(\psi, a_0) + (F)_0\right]\right\} d\psi = 0 \tag{2.7}$$

If a_0^* is a simple real root of the system (2.7), i.e. $\det(\partial R_2 / \partial a_0)^* \neq 0$, then for sufficiently small ε the system (1.3) admits a unique periodic solution, equal to a_0^* when $\varepsilon = 0$, which can be constructed in the form of a series or by means of consecutive approximations in whole powers of the parameter. When a is a scalar, then the periodic solution for the multiple real root a_0^* of (2.7) can be constructed in the manner analogous to that given above in Sect. 1°. The corresponding computations are bulky and therefore omitted here.

Investigation of the motions of an arbitrary order s $(s = 1, 2, \ldots)$ leads, in general, to the following defining system of equations:

$$R_1(a_0) = R_2(a_0) = \ldots = R_{s-1}(a_0) \equiv 0 \tag{2.8}$$

$$R_s(a_0) \equiv \lim_{\varepsilon \to 0} \varepsilon^{1-s} \int_0^{2\pi} \{f(a, \psi, \varepsilon) / [\omega(a) + \varepsilon F(a, \psi, \varepsilon)]\} d\psi = 0$$

which is assumed not to be identically satisfied with respect to a_0. Here we take the last system for a and substitute into it an expression in the form of a series in whole powers of ε, the coefficients of which are explicitly determined from the corresponding equations /1, 12/. This represents a simple case of higher order motions. If the system (2.8) admits a real root a_0^*, then simple and multiple solutions of higher order can be constructed in the manner analogous to that described above.

We shall note that a more general problem of constructing a periodic solution is of great interest when the system (2.8) is such that the rank of $(\partial R_s / \partial a_0)^*$ is less than n, the latter denoting the dimension of the vector a. Basically, the problem can be solved with help of the Poincaré method, in practice however the satisfactory construction of a solution is, in general, met with the difficulty of making the choice of the expansion.

The Liapunov stability of the simple higher order solutions is determined by the sign of the real parts of the roots of the characterisitic equation obtained with the corresponding accuracy: $\det\,[(\partial R_s / \partial a_0)^* - I\lambda] = 0$. If a is a scalar, then the periodic trajectory is asymptotically stable, and the solution of the system (1.2) is orbitally stable provided that the inequality

$$\int_0^{2\pi} (\omega + \varepsilon F)^{-2} \left[\frac{\partial f}{\partial a}(\omega + \varepsilon F) - f\left(\omega' + \varepsilon \frac{\partial F}{\partial a}\right)\right] d\psi < 0$$

holds an $\varepsilon > 0$ is sufficiently small.

3. Investigation of the selfrotational motions. Let us consider selfrotations of a system with one degree of freedom, almost conservative and representing a particular case of (1.1) /5/

$$x^{\cdot\cdot} + Q(x) = \varepsilon q(x, x^{\cdot}, \varepsilon) \tag{3.1}$$

Here x is the generalized coordinate, $x^{\cdot}$ is the velocity. It is also assumed that the functions Q and q are periodic in x with a constant period of 2π, and that the "potential energy" of the system

$$U(x) = \int_0^x Q(\varphi)\, d\varphi$$

is also a periodic function. In particular if $Q(x) = \nu^2 \sin x$ ($\nu = \text{const}$), we have a "pendulum" type system.

We shall only consider the rotational motions of the system (3.1), i.e. the motions for which $x^{\cdot} \geqslant \alpha > 0$ or $x^{\cdot} \leqslant \alpha < 0$. On the basis of the sign-definiteness of the phase trajectory $x^{\cdot} = y(x, \varepsilon)$ we can construct a rotational solution of the system (3.1) using the method developed in /5/. The period of such a motion can be found using quadratures

$$T(\varepsilon) \equiv \frac{2\pi}{\omega} = \int_0^{2\pi} \frac{dx}{|y(x, \varepsilon)|} \tag{3.2}$$

while the maximum and minimum values of the velocity are reached periodically for the values of x determined from equation $Q(x) = \varepsilon q(x, y(x, \varepsilon), \varepsilon)$.

To construct the required phase trajectory $y(x, \varepsilon)$ we use the integral equation (3.3) together with the necessary and sufficient condition of periodicity y (3.4)

$$\frac{y^2}{2} + U(x) = \varepsilon \int_0^x q(\varphi, y, \varepsilon)\, d\varphi + E_0 + v \tag{3.3}$$

$$\int_0^{2\pi} q(x, y, \varepsilon)\, dx = 0 \tag{3.4}$$

Here $E_0 = \text{const}$ is the unperturbed energy of the system, $v = \text{const}$, and $v = 0$ when $\varepsilon = 0$. In accordance with the Poincaré method we have

$$y = y(x,\ E_0,\ v,\ \varepsilon) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \varepsilon^i v^j y_{ij}(x,\ E_0), \tag{3.5}$$
$$y_{00} \equiv y_0 = \pm\sqrt{2}\,[E_0 - U(x)]^{1/2}$$

Substituting the expansion (3.5) into (3.3) and equating the coefficients of like powers of $\varepsilon^i v^j$, we find the required functions $y_{ij}(x, E_0)$. Substituting the expression $y(x, E_0, v, \varepsilon)$ into (2.4), we find the unknown parameter E_0 and $v = v(\varepsilon)$. When $\varepsilon = 0$ we have $v(0) = 0$ and the unperturbed energy must satisfy the equation

$$R_1(E_0) \equiv \int_0^{2\pi} q(x,\ y_0(x,\ E_0),\ 0) \equiv r(2\pi,\ E_0) = 0 \tag{3.6}$$

The case E_0^* being a simple real root of (3.6) satisfying the condition of rotation: $E_0^* > \max U(x)$ on $x \in [0, 2\pi)$, was studied in /5/. Let us consider the critical cases.

1°. Construction of the phase trajectories in the case of multiple roots. Let E_0^* be an admissible double root of (3.6). Then the periodic solution of (3.3) can be written in the form of series

$$y = \sum_{i=0}^{\infty} \delta^i y_i(x,\ E_0^*), \quad v = \sum_{i=1}^{\infty} \delta^i v_i \tag{3.7}$$

Here $\delta = \sqrt{\varepsilon}$ and for the coefficients of the series we obtain, in particular, $y_1 = v_1 / y_0$ where $v_1 = \pm\sqrt{b_1 / a}$, $a = (d^2R_1 / dE_0^2)^*$

$$b_1 = \int_0^{2\pi} \left[\left(\frac{\partial q}{\partial y}\right)_0 y_0^{-1} r(x) + \left(\frac{\partial q}{\partial \varepsilon}\right)_0\right] dx$$

Here and henceforth the dependence of r on the known E_0^* is not shown. Let us assume that $b_1 / a > 0$. Then all further coefficients v_i $(i \geqslant 2)$ are determined from the linear equations of the form $av_i = b_i$, where b_i are known constants. For example,

$$b_2 = \int_0^{2\pi} \left\{\left(\frac{\partial q}{\partial y}\right)_0 y_0^{-1} \left[\int_0^x \left(\frac{\partial q}{\partial y}\right)_0 y_0^{-1}\, d\varphi - \left(r(x) - \frac{v_1^2}{2} y_0^{-2}\right) y_0^{-2}\right] + \right.$$
$$\left. \left(\frac{\partial^2 q}{\partial y^2}\right)_0 \left(r(x) - \frac{v_1^2}{2} y_0^{-2}\right) y_0^{-2} + \left(\frac{\partial^2 q}{\partial y\, \partial \varepsilon}\right)_0 y_0^{-1}\right\} dx$$

The coefficients for $i > 2$ are computed in a similar manner. Thus if E_0^* is a double root and $b_1 / a > 0$, the phase trajectory splits by a quantity of the order of δ

$$y(x,\ \varepsilon) = y_0 + \sum_{i=1}^{\infty} (\pm\delta)^i y_i(x), \quad \delta = \sqrt{\varepsilon}$$

and we have no uniqueness in the accepted sense /1/.

If $b_1(E_0^*) = 0$, then all $y_{2j-1}(x) \equiv 0$ and $v_{2j-1} = 0$ $(j = 1, 2, \ldots)$. Then, as in Sect. 2, the expansion is carried out in whole powers of ε. The coefficient v_2 in the expansion (3.7) is given by the following quadratic equation:

$$av_2^2 + bv_2 + c = 0, \quad b = 2(db_1 / dE_0)^*$$

$$c = \int_0^{2\pi} \left\{ \left(\frac{\partial^2 q}{\partial y\, \partial \varepsilon} \right)_0 y_0^{-1} r(x) + \frac{1}{2} \left(\frac{\partial^2 q}{\partial y^2} \right)_0 r^2(x) y_0^{-2} + \frac{1}{2} \left(\frac{\partial^2 q}{\partial \varepsilon^2} \right)_0 + \left(\frac{\partial q}{\partial y} \right)_0 \left[\frac{1}{2} r(x) y_0^{-3} - y_0^{-1} \int_0^x \left(\left(\frac{\partial q}{\partial y} \right)_0 y_0^{-1} r(\varphi) + \left(\frac{\partial q}{\partial \varepsilon} \right)_0 \right) d\varphi \right] \right\} dx$$

The above equation has a solution when $b^2 - 4ac \geqslant 0$. In the case of strict inequality the equations for determining the further coefficients v_{2j} $(j \geqslant 2)$ will be linear and of the form $av_{2j} = c_j$. Further steps can be carried out in the analogous manner. The splitting in the case under consideration is of magnitude of the order of $\delta^2 = \varepsilon$, and again there is no uniqueness.

Let us briefly consider the case of a triple root, and explain the conditions under which the expansions have the form of (3.7) where $\delta = \sqrt[3]{\varepsilon}$. Substituting (3.7) and equating the coefficients we obtain, in particular, $y_1 = v_1 / y_0$, where v_1 is a real root of the equation $6d\, v_1^3 + b_1 = 0$, where $d = (d^3 R_1 / dE_0^3)^*$. For $b_1(E_0^*) \neq 0$ the further coefficients are determined from linear equations of the form $dv_i + g_i = 0$ $(i \geqslant 2)$, where g_i are found consecutively, e.g.

$$g_2 = \frac{1}{2} \int_0^{2\pi} \left\{ \frac{1}{6} \left(\frac{\partial^4 q}{\partial y^4} \right)_0 v_1^3 y_0^{-4} + 4 \left(\frac{\partial^2 q}{\partial y\, \partial \varepsilon} \right)_0 y_0^{-1} + \left(\frac{\partial^3 q}{\partial y^3} \right)_0 v_1^2 y_0^{-5} + \left(\frac{\partial^2 q}{\partial y^2} \right)_0 \left[4 y_0^{-2} r(x) - \frac{v_1^3}{2} y_0^{-6} \right] + 4 \left(\frac{\partial q}{\partial y} \right)_0 \left[y_0^{-1} \int_0^x \left(\frac{\partial q}{\partial y} \right)_0 y_0^{-1} d\varphi + \frac{v_1^3}{4} y_0^{-7} + y_0^{-1} r(x) + \frac{v_1^2}{2} y_0^{-5} \right] \right\} dx$$

The functions $y_i(x)$ are determined from (3.3), one after the other. In particular we have

$$y_2 = (v_2 - y_1^2 / 2) y_0^{-1}, \qquad y_3 = [r(x) + v_3 - y_1 y_2]\, y_0^{-1}$$

When $b_1^* = 0$, the expansions can be written in whole powers of ε or of $\sqrt{\varepsilon}$. The basic conclusion drawn from this investigation is, that multiple roots may cause splitting of the phase trajectories and solutions.

2°. Investigation of higher order selfrotations. Assume that we have the critical case in which the defining equation (3.6) is satisfied identically, i.e. independently of E_0. Then, using the Poincaré method we obtain from the expressions (3.5) and Eqs. (3.3) and (3.4) the following equation for E_0:

$$R_2(E_0) \equiv \int_0^{2\pi} \left[\left(\frac{\partial q}{\partial y} \right)_0 y_0^{-1} r(x, E_0) + \left(\frac{\partial q}{\partial \varepsilon} \right)_0 \right] dx = 0$$

If E_0^* is a simple real root satisfying the rotation condition, then the periodic phase trajectory can be uniquely determined in the form of a series in whole powers of ε. In fact, since $d^k R_1 / dE_0^k = 0$ for any E_0, we obtain equations which are linear in the unknown v_i $(i = 1, 2, \ldots)$, and these unknowns can be determined, just as y_i, consecutively from the equations $\alpha v_i + \beta_i = 0$, $\alpha = (dR_2 / dE_0)^* \neq 0$. In particular we have

$$\beta_1 = \int_0^{2\pi} \left\{ \left(\frac{\partial q}{\partial y}\right)_0 \left[-y_0^{-1} \int_0^x \left(\left(\frac{\partial q}{\partial y}\right)_0 y_0^{-1} r(\varphi) + \left(\frac{\partial q}{\partial \varepsilon}\right)_0 \right) d\varphi + \frac{1}{2} y_1^{-3} r^2(x) \right] + \frac{1}{2} \left(\frac{\partial^2 q}{\partial y^2}\right)_0 y_0^{-2} r^2(x) + \left(\frac{\partial^2 q}{\partial y\, \partial \varepsilon}\right)_0 y_0^{-1} r(x) + \frac{1}{2} \left(\frac{\partial^2 q}{\partial \varepsilon^2}\right)_0 \right\} dx = c$$

If $R_1 \equiv 0$ and $R_2 \equiv 0$ but $R_3(E_0) = \beta_1(E_0) \not\equiv 0$, then the problem of existence and uniqueness of the phase trajectory sought depends on the properties of the roots of the equation $\beta_1(E_0) = 0$. We treat this case just as we did the previous one. Generally speaking, a study of the phase trajectories and motions of arbitrary degree s leads to a system of defining equations of the type (2.8) where

$$R_s(E_0) \equiv \lim_{\varepsilon \to 0} \varepsilon^{1-s} \int_0^{2\pi} q(x, y, \varepsilon)\, dx = 0$$

and this equation is understood to have the meaning described in Sect. 2.

If E_0^* is a multiple root, then the multiple higher order selfrotations can be investigated in the same manner as the case of multiple roots for the first order selfrotations. The Liapunov stability of the phase trajectories at $\varepsilon > 0$ is sufficiently small and is defined by the inequality

$$\int_0^{2\pi} \frac{\partial q}{\partial y}\, dx < 0$$

For example, in the case of the simple higher order selfrotations the sufficient condition of orbital stability is, that the inequality $(dR_s / dE_0)^* < 0$ holds. In the case of multiple roots the stability is investigated exactly as in Sect. 3.

In conclusion we note that, using a known phase trajectory, we can employ the investigated approximations or expansion into series in powers of δ, construct the rotational solution of (3.1), compute the period (3.2) and other parameters of the steady motion /5/.

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ON THE THEORY OF STABILITY OF PROCESSES OVER A SPECIFIED TIME INTERVAL

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The present paper supplements and formulates in a more rigorous form, the statement of the problem on stability of processes over a specified time interval, which was given in /1, 2/. The refinement concerns the case in which the specified time interval is finite, and we find that an imposition of stronger constraints on the region of limiting deviations becomes necessary. As far as the character of the constraints imposed on the perturbations of the parameters of the process is concerned, the proposed formulation and the initial formulation are both related to /3/. We use the fact that a linear differential system can be transformed into a diagonal one, as the basis for establishing the necessary and sufficient conditions of stability of a linear process, and for obtaining certain conditions of stability and instability of a nonlinear process in the linear approximation. We show how transformation of a linear system to a "nearly" diagonal system can be utilized for the same purpose.

1. Choice of the region of limiting deviations. We introduce the region of limiting deviations using the class K_Δ^ω of ($n \times n$)-matrices $G(t) = (G_1 G_2 \ldots G_n)$ over the field of complex numbers, satisfying the following conditions on the interval $\Delta = [t_0, T)$, where T is a number greater than t_0, or ∞ : $\det G(t) \neq 0$ and the Hermitian norm of the columns $G_j(t)$ $(j = 1, 2, \ldots, n)$ coincides with a positive function $\omega(t)$, i.e. $\| G_j(t) \| = \sqrt{(G_j, G_j)} = \omega(t)$.

The region of limiting deviations is defined as follows: